Cities and Space: Common Power Laws and Spatial Fractal Structures

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Power Laws and Fractal Structure

- City size distributions are known to be well approximated by power laws across a wide range of countries.

$$Pr(S > s) \approx cs^{-\alpha}, s \to \infty.$$

- Fractal structure: smaller parts of a system structurally resemble the larger ones, including the entire system (Mandelbrot 1982).

- An implication of fractal structure: If a fractal structure exhibits a power law as a whole, then its smaller parts must also exhibit similar power laws.

- If a system exhibits this similarity property, the system is said to exhibit a **common power law** (**CPL**).

CPL and Fractal Structures: Spatial or Not?

- Gibrat's law: growth rate of individual cities are *identically and independently distributed*.
- Under various mechanisms (e.g., Gabaix 1999; Luttmer 2012), Gibrat's law leads to a power law.
- Any subset of cities must also have this *i.i.d.* property (**fractal structure**) ⇒ power laws should hold in any sufficiently large subset set of cities, i.e., **CPL**.
- But the *i.i.d.* assumption is so inclusive that a CPL must hold for arbitrary subsets of cities, regardless of the spatial relations between them.
- Does space matter?
- Think of central place theory (Christaller 1933; Fujita, Krugman and More 1999; Tabuchi and Thisse 2011; Hsu 2012; Hsu, Holmes and Morgan 2014).
- A central place hierarchy is a **spatial fractal structure** and exhibits a **spatial CPL**.

What This Paper Does — Spatial Grouping Property

Using data from the US, France, Germany, Japan, China, and India, we show evidence for the *spatial grouping property*: larger cities tend to serve as centers around which smaller cities are grouped.

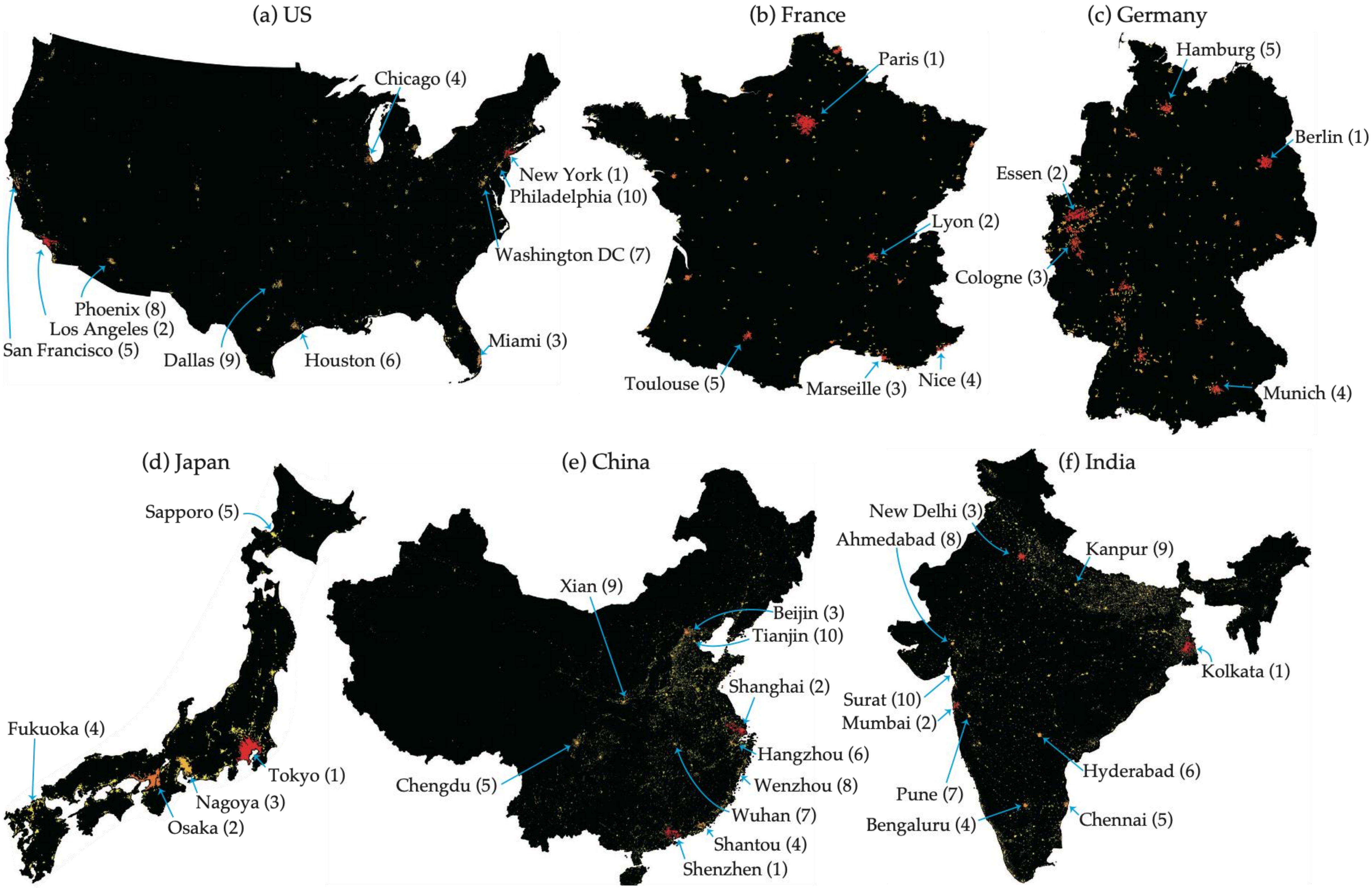
- A test on the *spacing-out property*: large cities are much more spaced out than a random pattern would predict.
- Corroborating evidence on inter-city trade flows.
- Spatial grouping is naturally be **recursive**.

What This Paper Does — Spatial CPL Test

Based on the spatial grouping property, we partition cities in each country as a spatial hierarchical partition.

- There is a strong graphical CPL pattern across different cells of the spatial hierarchical partition.
- Moreover, the degree of commonality of the power laws are statistically more significant than (aspatial) random partitions, establishing a spatial CPL.

A city = A contiguous area with at least 1,000 people/km2 and total population 10,000.



SPACING-OUT TEST

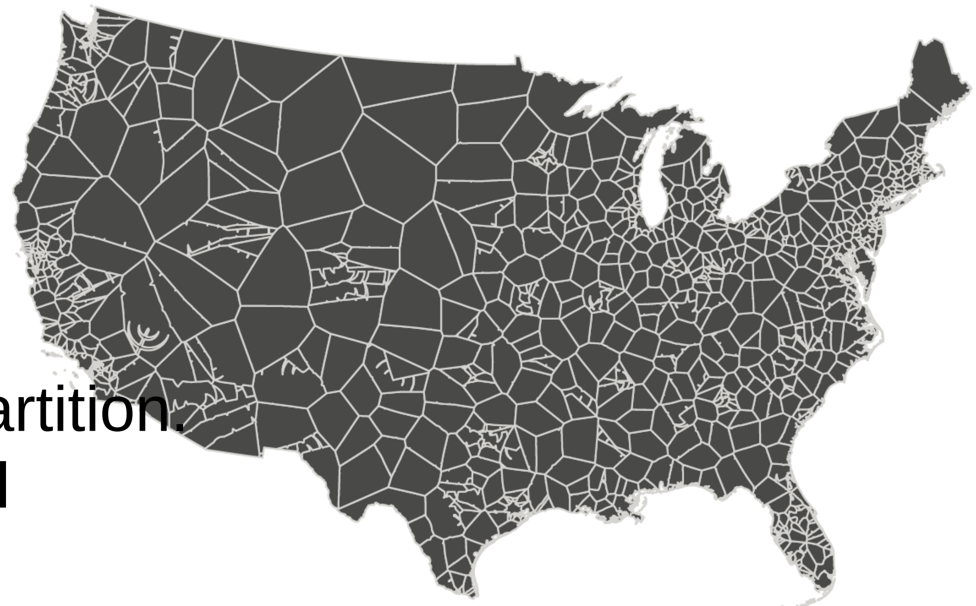
Idea of the test

- Topographical factors (e.g., Rocky Mt.)
- Historical factors (e.g., transport technology)

➔ Different spatial frequency of cities
in different parts of the country.

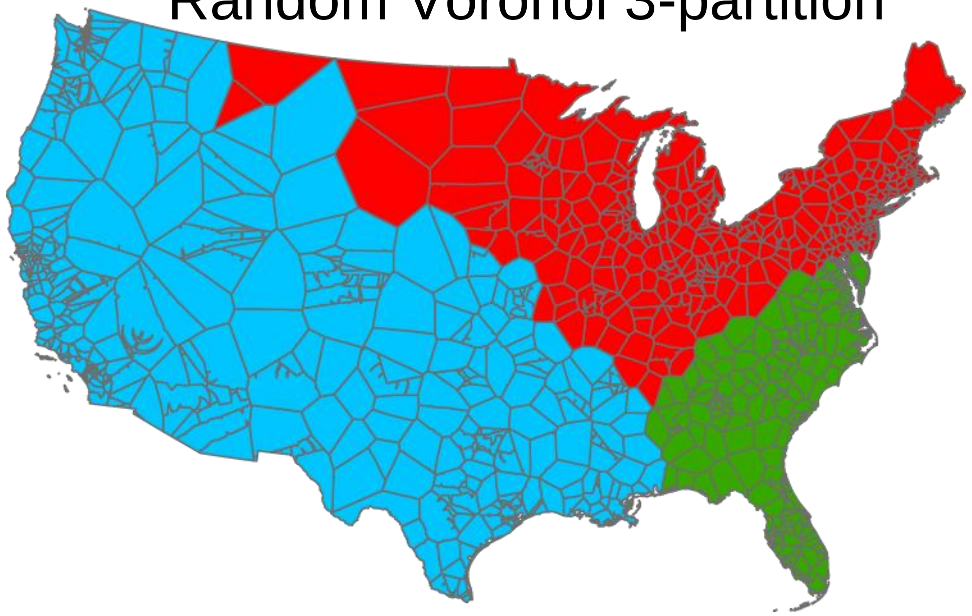
➔ Randomly draw K cities and form a Voronoi K-partition.
The draws are essentially conditioned on the spatial frequency of cities.

Voronoi partition of space w.r.t. all cities



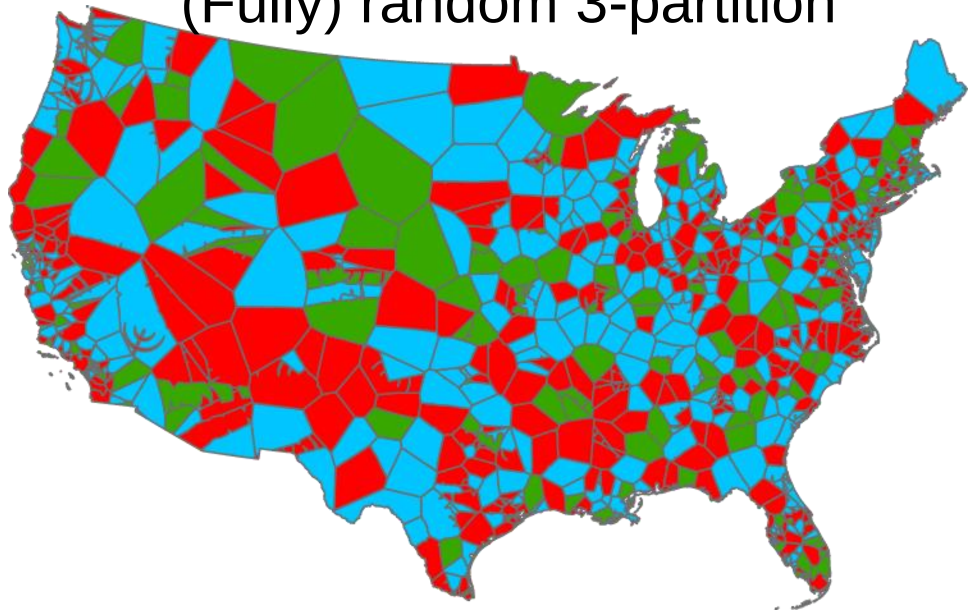
Compare the count of cells containing the largest L cities

Random Voronoi 3-partition



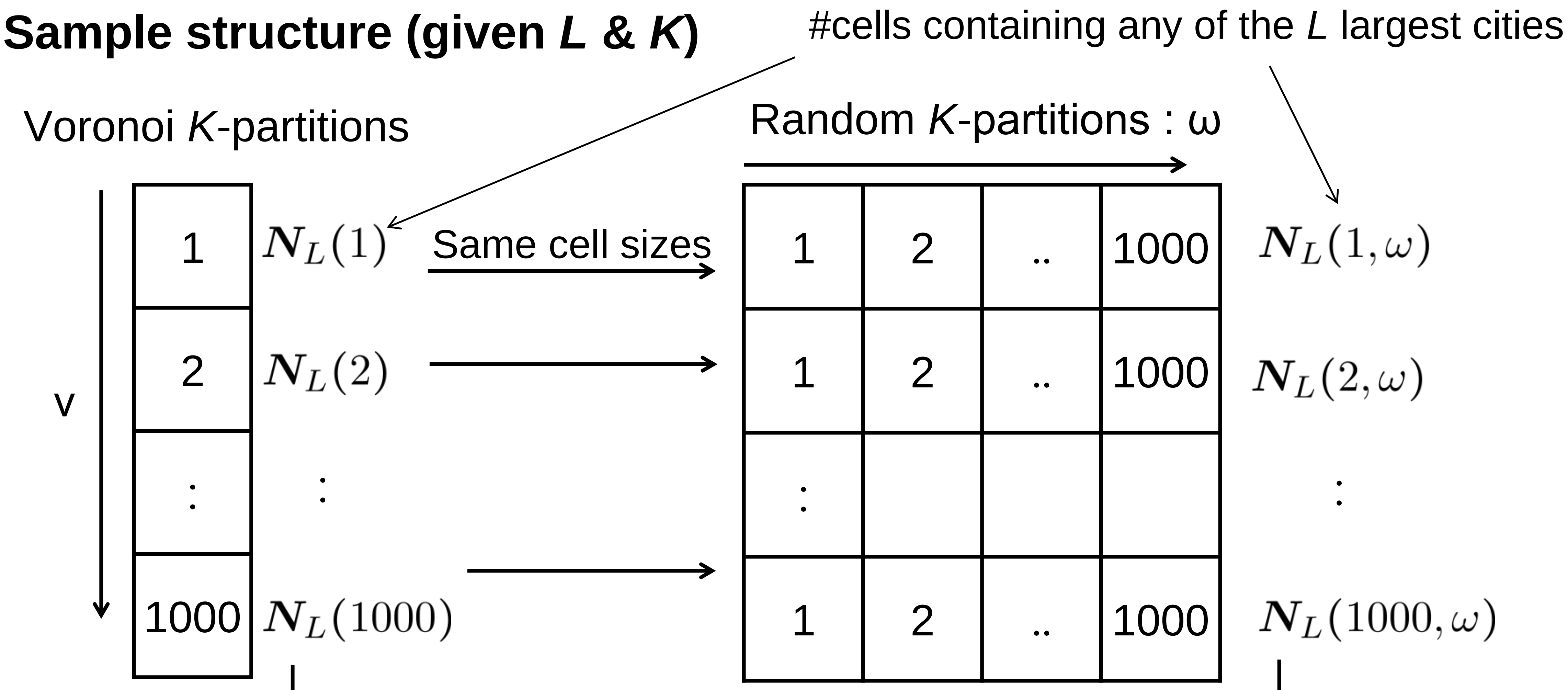
East cells tend to be smaller

(Fully) random 3-partition



East cells are picked more frequently

Sample structure (given *L* & *K*)

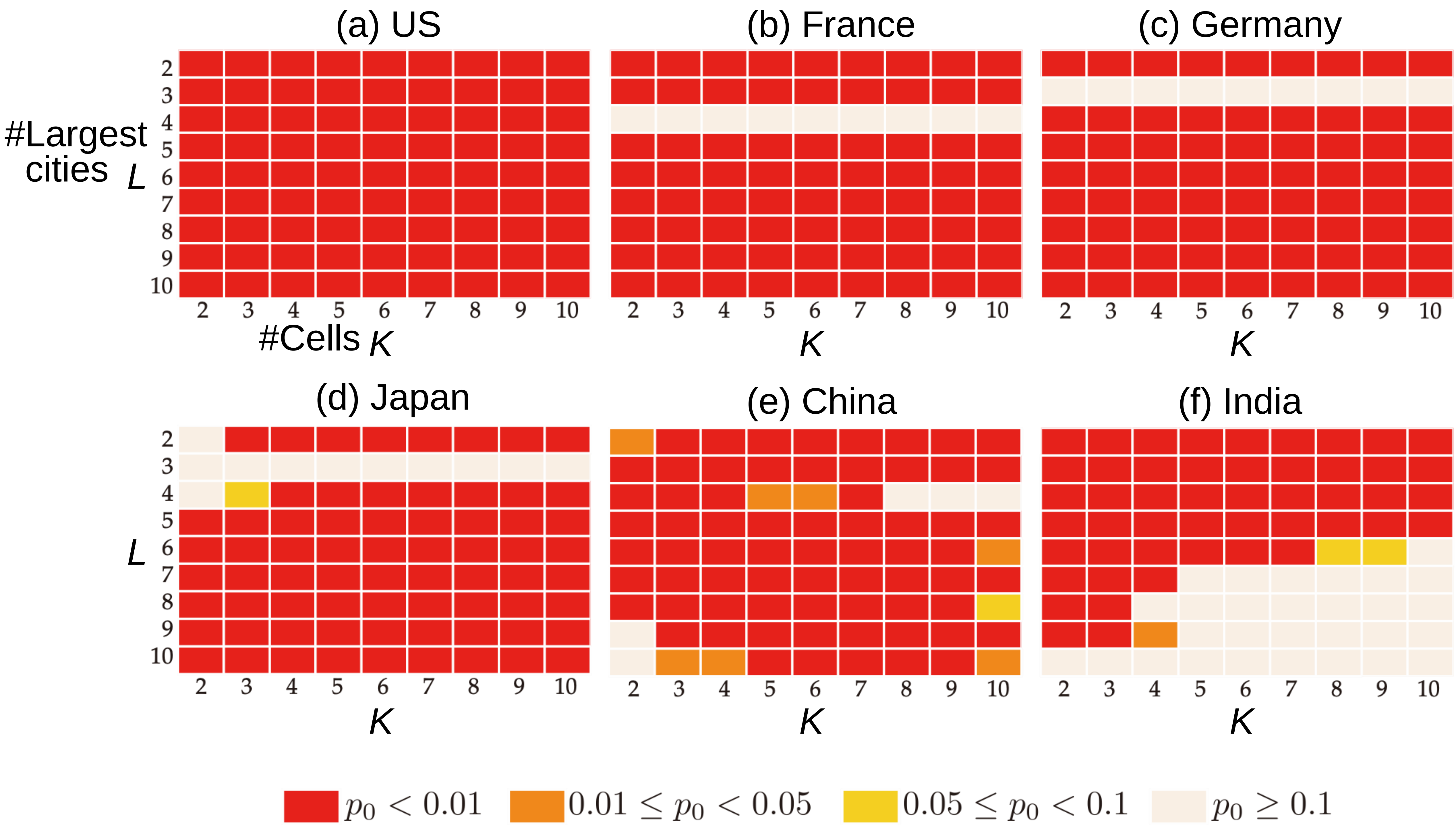


$$\overline{N}_L = \frac{1}{1000}\sum_{v=1}^{1000} N_L(v)$$

$$\overline{N}_L(\omega) = \frac{1}{1000}\sum_{v=1}^{1000} N_L(v,\omega)\ ,\quad \omega = 1,\ldots,1000$$

H_0 : $\overline{N}_L$ and $\overline{N}_L(\omega)$ come from the same population.

$$p\text{-value} = \frac{\#\{\overline{N}_L(\omega) \geq \overline{N}_L : \omega = 1,\ldots,1000\} + 1}{1001}$$


(a) US
(b) France
(c) Germany
(d) Japan
(e) China
(f) India
#Largest cities L
#Cells K
2 3 4 5 6 7 8 9 10
$p_0 < 0.01$
$0.01 \leq p_0 < 0.05$
$0.05 \leq p_0 < 0.1$
$p_0 \geq 0.1$

SPACING-OUT TEST — CONTINUED

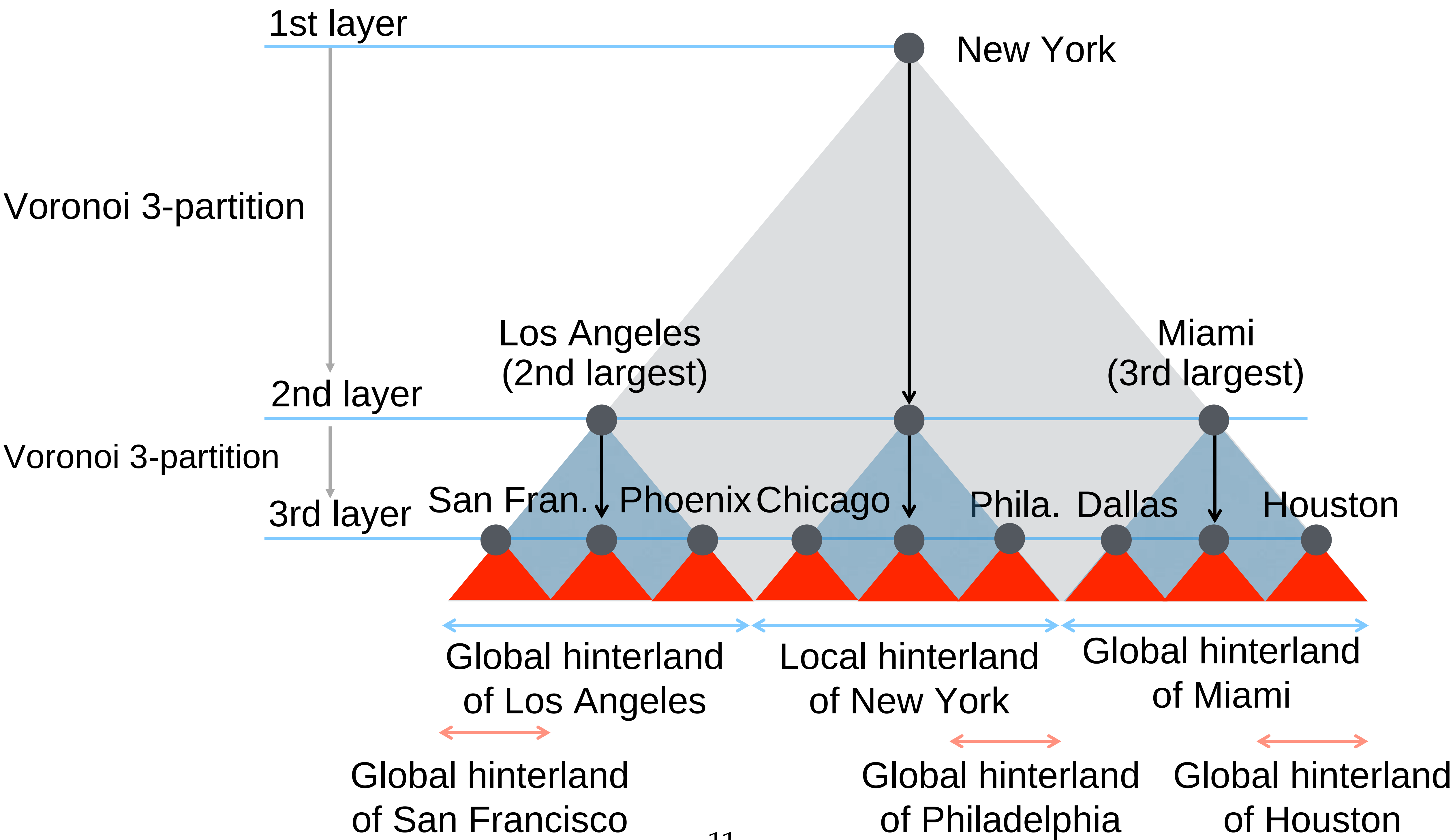
- Largest cities are generally much more spaced out than a random partition would predict.

- Why are cities spaced out? In central place theory: agglomeration and spatial competition are key elements.

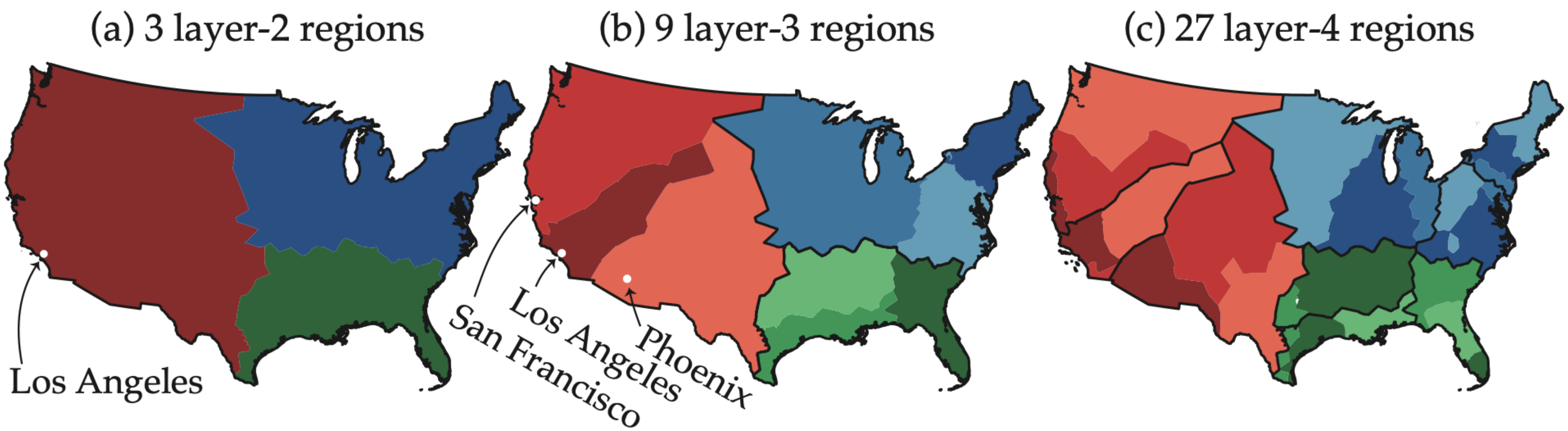
- But natural advantages may also matter.
  - In France, the third and fourth largest cities (Marseille and Nice) are rather close.
  - In Germany and Japan, the second and third largest cities (Essen and Cologne; Osaka and Nagoya) are rather close.
  - In India, the spacing out test are significant only when $L \leq 6$; given the level of development in India, natural advantages may matter more there.

Spatial Hierarchical *L*-partition of Cities

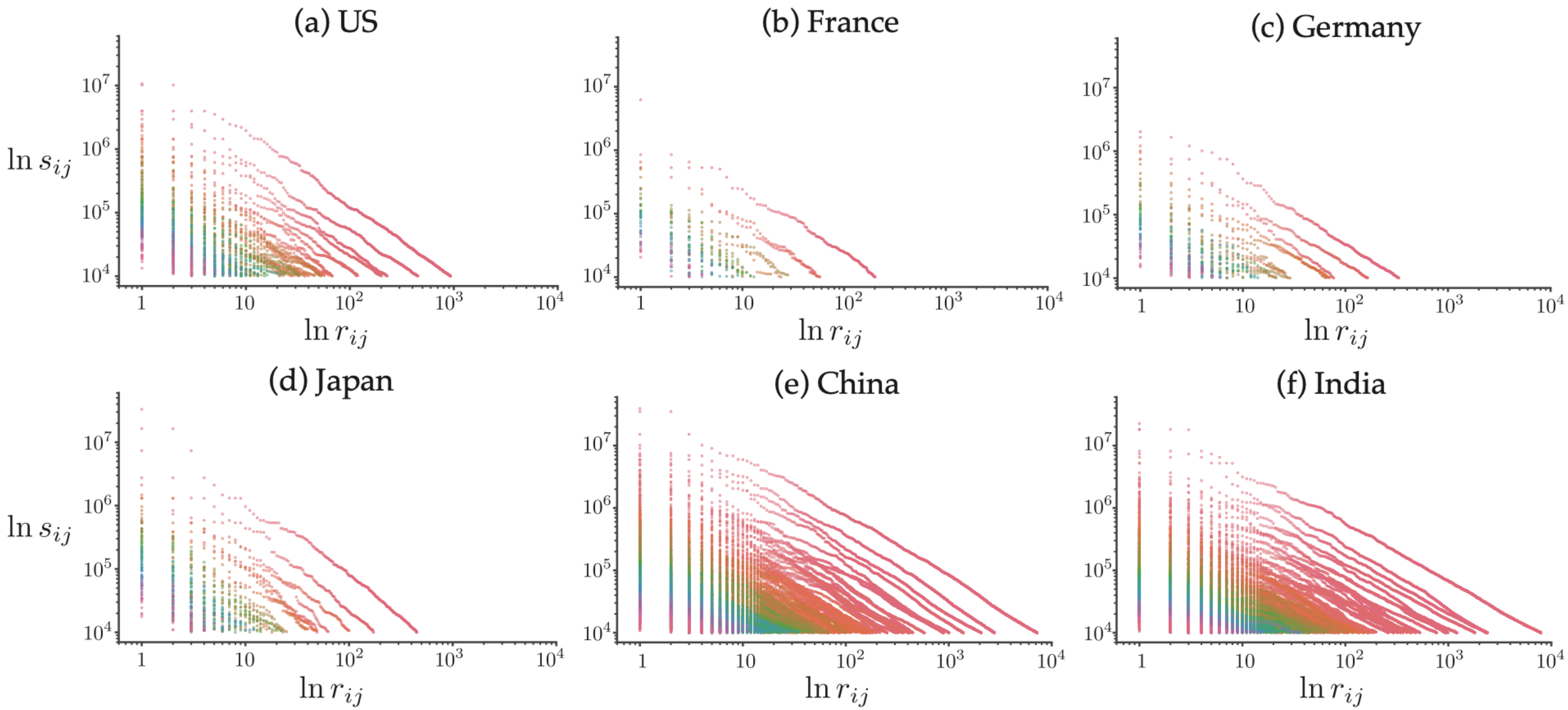
Example. $L = 3$ **for the US**



Example. $L = 3$ **for the US**



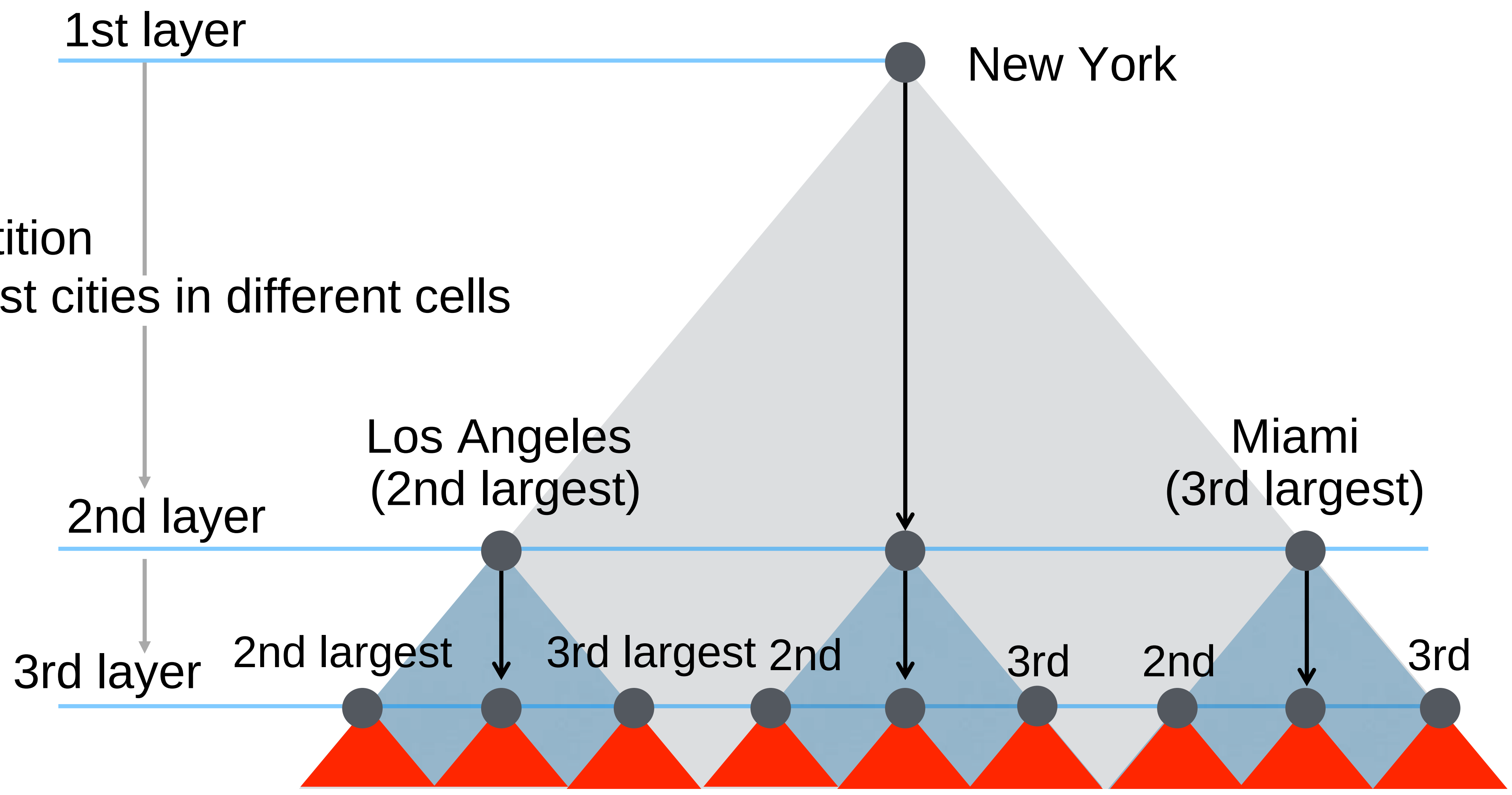
Example. Hierarchical 3-partition



Random counterfactual partitions :

Hierarchical *L*-partitions **without** spatial relations among cities

Random 3-partition
with the 3 largest cities in different cells



Categorical regression model :

Pop. size of city *i* — Size rank of city *i* in cell *j*

$$\ln s_{ij} = b_1 - \theta \ln(r_{ij} - 0.5) + \sum_{h=2}^{m} \beta_j \delta_j(h) + \varepsilon_{ij}$$

Common power coeff. (θ)

Gabaix-Ibragimov adjustment ($r_{ij} - 0.5$)

Cell dummy ($\delta_j(h)$) for layers 2 and lower

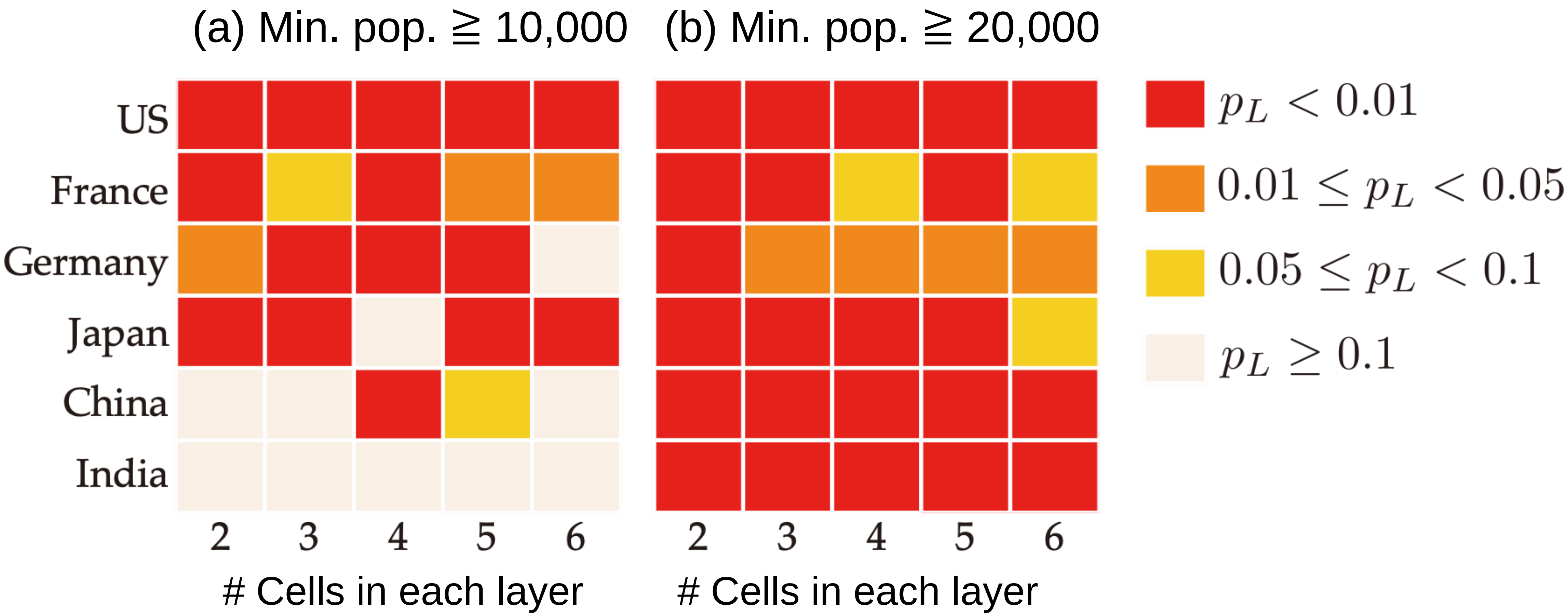
➔ Intercept for cell *j* : $b_j = b_1 + \beta_j$

Measure of fit :

$$\text{RMSE} = \sqrt{\frac{1}{\sum_{j=1}^{m} n_j} \sum_{j=1}^{m} \sum_{i=1}^{n_j} \left(\ln s_{ij} - \widehat{\ln s_{ij}} \right)^2}$$

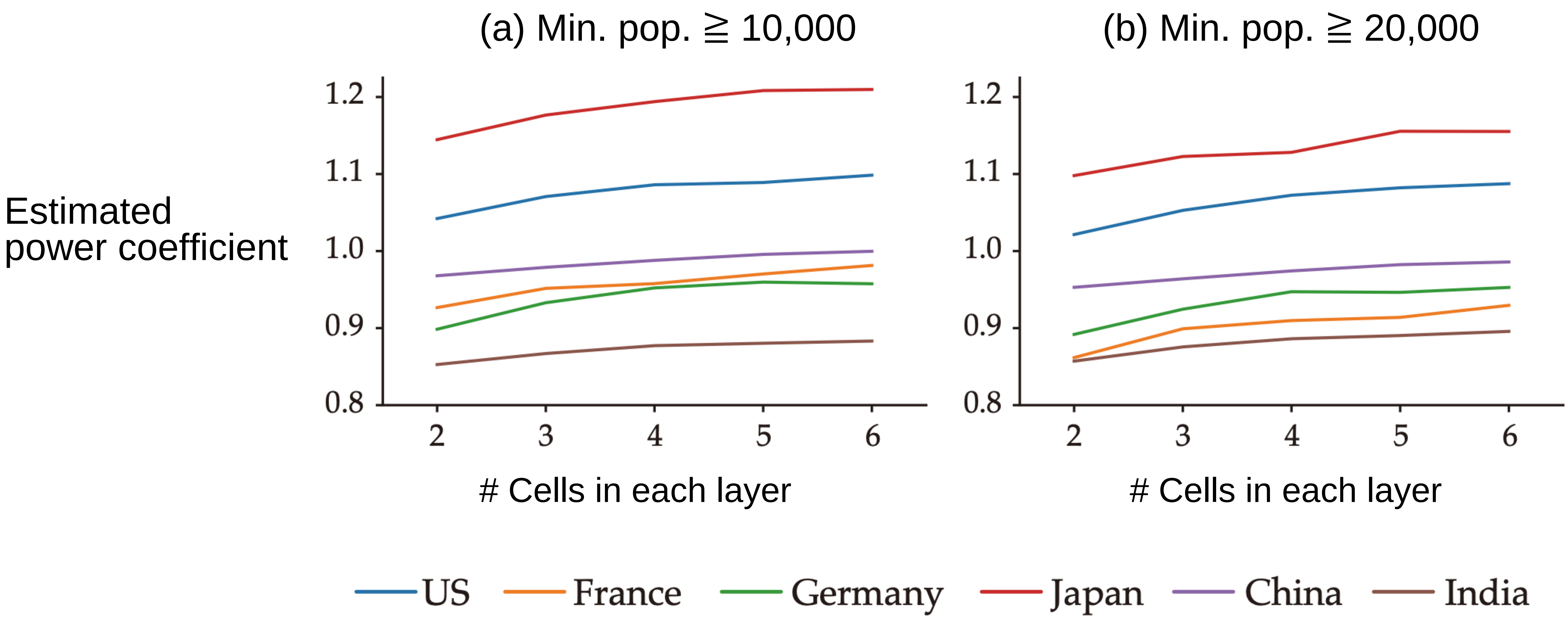
H_0 : The actual and counterfactual RMSE values come from the same population.

$$p\text{-value} = \frac{\#\{\text{RMSE}_L(\omega) \leq \text{RMSE}_L \ : \ \omega = 1, \dots, 1000\} + 1}{1001}$$



- Population density is generally higher in China and India.
- The definition of a city (population density $\geq 1000/km^2$ and a total of $\geq 10{,}000$) works well for the four developed countries, but less so for China and India because many rural villages are dense and large enough to quality as cities (e.g., Ganges Basin).
- Population: China and India are more than 4 times larger than the US, but the # of cities: US (931), China (7,204), India (7,915)
- Increase the threshold to $\geq 20{,}000$

CROSS-COUNTRY COMPARISON



※There is no CPL across countries.

※This is consistent with the idea of spatial fractal structure because the examined countries are geographically separated (except France and Germany), so that there are generally no clear spatial hierarchical relations among them.

Inter-City Trade Flows and Spatial Grouping Property

- This paper shows evidence for the spacing-out property, which in turn supports the spatial grouping property.
- In the Appendix, we use inter-city trade data from Japan that is aggregated from individual shipments by manufacturing establishments.
- We find that in a spatial hierarchical partition, centers and their hinterland cities trade with each other much more frequently than with other cities.
- For example, in a spatial hierarchical 3-partition,
  - a layer-3 **center** on average **exports** to **its own hinterland cities** 15.4 times more than to hinterland cities of other same-layer centers;
  - a layer-3 **hinterland city** on average **imports** 175.3 times more from **its own center** than from other same-layer centers.

Concluding Remarks

- Random growth theories imply that the CPL should hold for our random counterfactuals.
- Our test results: the CPL is much stronger for spatial hierarchical partitions of cities than for random subsets, and thus cast doubt on the random growth explanation.
- This **spatial CPL** result suggests the existence of a spatial fractal structure.
- Central Place Theory:
  - the degree of scale economies differs across goods and hence that the spatial extent of markets also differs.
  - Given the existence of certain agglomeration forces and competition mechanisms, a hierarchy of cities (and hence a city size distribution) naturally arises.
  - Christaller's (1933) structure already suggests a spatial fractal structure. Beckmann (1958) show that power laws can emerge from a central place hierarchy.
  - By building an equilibrium model of firm entry with a continuum of goods and a continuum of geographic space, Hsu's (2012) central-place hierarchy yields an explicit spatial fractal structure which exhibits a spatial CPL.

Concluding Remarks

- Other potential approaches:
  - a more **fractal-theoretical approach** as in Batty and Longley (1994)
  - Extensions of **random growth processes by adding spatial relations** among cities (Eaton and Eckstein 1997, Rosenfeld et al. 2008, Rybski et al. 2013)
  - Adopting techniques from **spatial networks** (Batty and Longley 1994, Gastner and Newman 2006, Clauset et al. 2008, Batty 2013, Berliant and Watanabe 2018)

- Whether and how these approaches might generate spatial CPL are questions yet to be investigated.